Characterization of Periodic Attractors in Neural Ring Networks *

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Abstract

The paper presents a discussion of parameterized discrete dynamics of neural ring networks. For specific parameter domains stable periodic orbits coexist. Their periods and the number of orbits of a given period are determined. Even n-rings, i.e. rings with an even number of inhibitory connections, exhibit mainly stable period-n orbits. Odd n-rings display mainly stable period-2n orbits. The dynamical effects of inhibitory connections are analysed, and a characterization of attractors in terms of their "firing pattern" is presented.

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1 Introduction

To date, oscillatory dynamics of neural networks is considered less important with respect to information processing applications than convergent dynamics. In fact, most models of artificial neural networks, like e.g. Hopfield type networks (Hopfield, 1982) or feedforward networks (Rumelhart, Hinton & Williams, 1986), have convergent activation dynamics (Hirsch, 1989).

The oscillatory dynamics of recurrent networks of reasonable size is in general difficult to study, because mathematical tools for the analysis of periodic orbits in high dimensional phase spaces are not yet sufficiently developed. The fact, that the network dynamics can be represented as a parametrized dynamical system on an activation phase space seems to be of no help because of the large number of (more or less) relevant control parameters. Nevertheless, parameter studies of discrete neurodynamics have been shown to be fruitful for the understanding of generic dynamical aspects of recurrent networks. This was demonstrated for example in Renals & Rohwer (1990), Paulus, Gass & Mandell (1989), Chapeau-Blondeau & Chauvet (1992), and Blum & Wang (1992).

In this paper we will analyse the parameterized discrete dynamics of recurrent n-neuron networks with only one loop, called n-ring networks. Basic aspects of the ring dynamics are described in Section 2. For a detailed analysis we distinguish between even and odd n-ring networks; i.e. the number of inhibitory weights in these networks is even and odd, respectively. In both cases bifurcations from a stable fixed point to a whole set of coexistent stable periodic orbits do appear. For special n-rings this was described already in Blum & Wang (1992). Here we give a detailed analysis of these coexistent attractors and their properties.

Section 3 is devoted to simulational studies of homogeneous ring networks, for which the basic features of n-rings can already be observed. For general n-rings a mathematical analysis of the dynamical effects of inhibitory connections is given in Section 4. Introducing formal activity sequences the mathematical analysis is continued in Section 5 to derive the number of coexisting periodic attractors in n-rings. The mathematical results were of course suggested by simulations of n-rings performed with a program developed by the author.

Especially odd n-ring networks could be of value for information processing techniques. This is due to the high number of coexisting periodic attractors (approximately $2^n/2n$ according to Corollary 1 in Section 5), which may be used for information storage or coding. That stable limit cycles can be used for data storage was shown e.g. in Andreyev, Dmitriev, Chua & Wu (1992), Baird (1986), Li & Hopfield (1989). Using an n-ring as a "middle layer" network, i.e. "sandwiching" it with appropriate input and output networks, memories with high storage capacity could be realizable. In Section 6 the results obtained for discrete dynamics are discussed and compared with those for the continuous-time case.

2 Neural n-ring networks

The standard additive nonlinear neuron model is chosen, i.e. the activity a_i of unit i is given by the sum over the weighted outputs o_j of units j connected to unit i plus a bias term θ_i:

$$a_i := \sum_{j=1}^{n} w_{ij} o_j + \theta_i \quad , \tag{1}$$

where w_{ij} denotes the weight from unit j to unit i. The output $o_i := \sigma(a_i)$ of unit i is given here by the sigmoidal transfer function

$$\sigma(a) := \frac{1}{1 + e^{-a}} \quad . \tag{2}$$

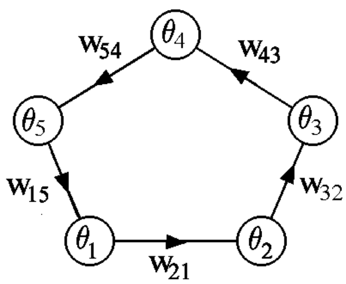

Figure 1: A five neuron ring network (5-ring).

We will consider the dynamics of n neurons belonging to a unique directed loop of connections. This structure (compare Figure 1) is called a neural n-ring network or *n-ring* for short. The network has a cyclic weight matrix, and its discrete dynamics is given by

$$a_i(t+1) := w_{ii-1} \sigma(a_{i-1}(t)) + \theta_i \ , \ a_0 := a_n \ , \ w_{10} := w_{1n} \ , \ i = 1, \ldots, n \ . \tag{3}$$

We may represent this dynamics as a 2n-parameter family of maps $f : \mathbf{R}^n \to \mathbf{R}^n$

$$a \mapsto f(a; \rho) \quad , \quad a \in \mathbf{R}^n \quad , \quad \rho = (\theta_1, \ldots, \theta_n, w_{1n}, \ldots, w_{nn-1}) \in \mathbf{R}^{2n} \quad . \tag{4}$$

The 2n control parameters are: the connections w_{ii-1} between the units, and the parameters θ_i, $i = 1, \ldots, n$, which are interpreted here as the sum of the fixed internal bias $\overline{\theta_i}$ and the total stationary external input I_i of unit i. For convenience we define the following quantities with respect to a given n-ring:

$$A := \prod_{i=1}^{n} w_{ii-1} \quad , \quad S(a) := \prod_{i=1}^{n} \sigma'(a_i) \quad , \quad a \in \mathbf{R}^n \quad . \tag{5}$$

If the number of inhibitory connections is even (odd), i.e. $A > 0$ ($A < 0$), the n-ring is called an *even (odd) n-ring.*

In general, transfer functions are assumed to be sigmoids, i.e. bounded, monotone increasing functions σ, $\sigma' > 0$, such that there exists a unique value $x \in \mathbf{R}^1$ where $\sigma'(x)$ attains a local (and global) maximum. Define the constant $\alpha > 0$, depending on the choice of the transfer function σ, to be this maximum, i.e.

$$0 < \sigma'(x) \leq \alpha \quad , \quad x \in \mathbf{R}^1. \tag{6}$$

For an n-ring we then have

$$0 < S(a) \leq \alpha^n \quad , \quad a \in \mathbf{R}^n \quad . \tag{7}$$

For the sigmoid (2) of our model we have $\alpha := \sigma'(0) = 1/4$.

A fixed point (a^*, ρ^*) of the dynamics (3), i.e. $f(a^*, \rho^*) = a^*$, is asymptotically stable, if the linearization $D_a f(a^*, \rho^*)$ of f at (a^*, ρ^*) has eigenvalues λ_k satisfying $|\lambda_k| < 1$, $k = 1, \ldots, n$. Then, with Equation (5), the eigenvalues at a^* for the case $A > 0$ can be given in the form

$$\lambda_k(a^*) = (AS(a^*))^{1/n} e^{i\pi 2(k-1)/n} \quad , \quad k = 1, \ldots, n \quad , \tag{8}$$

and for the case $A < 0$ by

$$\lambda_k(a^*) = (|A|S(a^*))^{1/n} e^{i\pi(2k-1)/n} \quad , \quad k = 1, \ldots, n \quad , \tag{9}$$

where the factor exponentials correspond to the complex n-th root of $+1$ and -1, respectively. Thus, the stability condition for a fixed point $(a^*; \rho^*)$ becomes

$$|A \cdot S(a^*)| < 1 \quad . \tag{10}$$

Of course, situations where a fixed point a^* becomes unstable are dynamically most interesting. At these nonhyperbolic fixed points at least one of the eigenvalues $\lambda_k(a^*)$ has modulus 1. For the system (3) considered here we see from Eqs. (8) and (9), that if one eigenvalue has modulus 1, then all eigenvalues $\lambda_k(a^*)$, $k = 1, \ldots, n$ have modulus 1. With $\alpha = 1/4$, the origin $a^* = 0$ is nonhyperbolic iff

$$|A| = \alpha^{-n} = 4^n \quad , \quad 2\theta_i = -w_{ii-1} \quad , \quad i = 1, \ldots, n \quad . \tag{11}$$

To analyse which type of bifurcation occurs at these nonhyperbolic fixed points we distinguish between even and odd rings. The bifurcation set $B^+ \subset \mathbf{R}^{2n}$ ($B^- \subset \mathbf{R}^{2n}$) denotes the set of parameter values ρ for which the dynamics (3) has a nonhyperbolic fixed point $a^*(\rho)$, what for this systems means that all n eigenvalues satisfy $|\lambda_k(a^*(\rho))| = 1$. The bifurcation sets B^+ and B^- correspond to even and odd rings, respectively. The domains enclosed by the bifurcation sets B^+ and B^- will be denoted by D^+ and D^-. They are called *hysteresis* (D^+) and *oscillatory* (D^-) domains for reasons which will become clear from the next section.

3 Homogeneous n-ring networks

To get an idea how these bifurcation sets look like, we will first simulate *homogeneous* n-rings for which $w_{ii-1} = w$, $i = 1, \dots, n$, and we also choose homogeneous bias/input terms, i.e. $\theta_i = \theta$, $i = 1, \dots, n$. The typical situation for this two parameter dynamics is shown in Figure 2 for an even 8-ring.

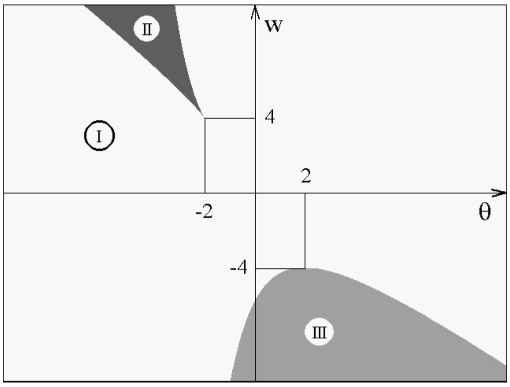

Figure 2: Bifurcation set B^+ (= border of domains II and III) in the (θ, w)-parameter subspace of an homogeneous even 8-ring.

For an excitatory (or *cooperative*) 8-ring ($w > 0$) we have a bifurcation set B^+ (the border of domain II) with a typical cubic singular point at $(\theta_c, w_c) = (-2, 4)$. In fact, the bifurcation set is identical with that of a single self-excitatory unit, as described in (Pasemann, 1993a), where it was shown, that the singular point corresponds to a cusp catastrophe. This suggests that also the dynamics of an excitatory n-ring is governed by a cusp catastrophe potential. Here, at bifurcation points of B^+, we observe bifurcations from a stable fixed point to a set of coexisting stable periodic orbits: In Section 5 we will prove analytically that there are 30 stable orbits of period eight, three stable orbits of period four, one stable orbit of period two, and two stable fixed points.

The bifurcation set B^+ (the border of domain III in Figure 2) of an inhibitory (or *competitive*) 8-ring ($w < 0$) has a typical quadratic singular point at $(\theta_c, w_c) = (2, -4)$. Recall, that although the 8-ring is inhibitory, it is an *even* 8-ring. Again, this bifurcation set is identical with that of a single unit with inhibitory self-connection w (Pasemann, 1993a). At points of B^+ we observe the same type of bifurcations as for the excitatory 8-ring described above. Although the number of stable orbits of period one, two, four and eight is the same, the orbits themselves are different (compare Section 5).

Repeating this simulation for a homogeneous 9-ring we derive exactly the same bifurcation sets B^+ and B^- and singular points (θ_c, w_c). For the excitatory 9-ring ($w > 0$) we observe at points of B^+ the same type of bifurcations as for the corresponding 8-ring: from a stable fixed point to stable periodic orbits, now mainly of period $n = 9$, and two stable fixed points. More explicitly, as is demonstrated in Section 5, there are 56 period-9 and two period-3 orbit attractors coexisting with two fixed point attractors.

The inhibitory 9-ring ($w < 0$) is an *odd* ring. The bifurcations at points of B^- are of different type. Here we observe bifurcations from a stable fixed point to stable orbits, mainly of period 18. In fact, in the following sections we demonstrate analytically that there are no stable fixed points but 28 period-18, one period-6 and one period-2 orbit attractors, all coexisting (compare Section 5).

These simulation results for homogeneous even and odd n-rings hold true for all n. (In fact, they hold true for general n-rings too as we will see in the next sections.) With respect to time varying inputs even and odd networks will behave quite differently. Consider first an even homogeneous n-ring with $|w| > 4$, i.e. the system exhibits two stable fixed points. If θ is varied slowly back and forth across the bifurcation set B^+, i.e. in such a way that the system is always able to approach an attractor, then a hysteresis effect will occur. If we start with θ outside the domain D^+, the stable periodic orbits existing for $\theta \in D^+$ will never be seen. The system behaves simply as a bistable system (as long as there is no noise introduced).

For an odd homogeneous n-ring with $w < -4$ the same procedure, i.e. varying θ back and forth over the bifurcation set B^-, will result in the appearance and disappearance of periodic behaviour. Exactly one of the various periodic attractors will be observed during this process. This periodic attractor is the one with a periodic point next to the, now unstable, fixed point. Here an n-ring behaves like a controllable oscillator.

The same simulational observations are made in the general case if we vary the bias/input vector $(\theta_1, \ldots, \theta_n)$ along a path crossing the corresponding bifurcation sets B^+ or B^-. Therefore the domains D^+ and D^- in $\mathbf{R}^n$ are called the hysteresis and the oscillatory domain.

4 Effects of inhibitory weights

To study the dynamical effects of inhibitory weights in an n-ring (3) we analyse the system in a different form. For any parameter vector $\rho \in \mathbf{R}^{2n}$ there exists at least one fixed point of (3), which will be denoted by $\bar{a} = \bar{a}(\rho)$; its components satisfy

$$\bar{a}_i = w_{ii-1}\, \sigma(\bar{a}_i) + \theta_i \quad , \quad i = 1, \ldots, n \quad .$$

For fixed ρ, we then introduce new system coordinates v_i by

$$v_i := a_i - \bar{a}_i \quad , \quad i = 1, \ldots, n \quad . \tag{12}$$

Defining $h_i := sign(w_{ii-1})$, i.e. $h_i \in \{+1, -1\}$ and writing

$$w_{ii-1} = h_i\, w_i \quad , \quad w_{10} = w_{1n} \quad , \quad w_i > 0 \quad , \quad i = 1, \ldots, n \quad ,$$

we introduce a transformed sigmoidal function g_{ρ_i}, with $\rho_i := (w_{ii-1}, \theta_i) \in \mathbf{R}^2$, by

$$g_{\rho_i}(x) := w_i \cdot (\sigma(x + \bar{a}_i) - \sigma(\bar{a}_i)) \quad , \quad x \in \mathbf{R} \quad , \quad i = 1, \ldots, n \quad . \tag{13}$$

Since σ is strictly increasing the function g_{ρ_i} satisfies

$$x \cdot g_{\rho_i}(x) \geq 0 \quad , \quad x \in \mathbf{R} \quad . \tag{14}$$

A short calculation then shows that the system $v(t+1) = G_\rho(v(t))$ given in component form by

$$v_i(t+1) = h_i \, g_{\rho_i}(v_{i-1}(t)) \quad , \quad v_0 = v_n \quad , \quad i = 1, \ldots, n \quad , \tag{15}$$

is equivalent to system (3). Trivially, the origin is always a fixed point of the system G_ρ which is globally stable for $|A \cdot S(\bar{a}(\rho))| < 1$, and it is unstable for $|A \cdot S(\bar{a}(\rho))| > 1$, i.e. for $\rho \in D^+$ or $\rho \in D^-$. Defining the sigmoidal function $\hat{g}_i$ by

$$\hat{g}_i := g_{\rho_i} \circ g_{\rho_{i-1}} \circ \cdots \circ g_{\rho_{i+1-n}} \quad , \quad i = 1, \ldots, n \quad .$$

a simple calculation gives

$$v_i(t+n) = h \, \hat{g}_i(v_i(t)) \quad , \quad h := \prod_{i=1}^{n} h_i \quad , \quad i = 1, \ldots, n \quad . \tag{16}$$

Of course $h = +1$ ($h = -1$) iff the n-ring is even (odd). We now can proof the following

Lemma 1 *If v^* is a periodic point of an n-ring (15), then each component v_i^* is a fixed point of the sigmoidal function $\hat{g}_i$, $i = 1, \ldots, n$.*

Proof: Suppose v^* is an r-periodic point of G_ρ for some $r \geq 1$. Then $G_\rho^r(v^*) = v^*$, and also $G_\rho^{rn}(v^*) = v^*$. For the components this implies

$$v_i^*(t) = h^r \hat{g}_i^r(v_i^*(t)) \quad , \quad i = 1, \ldots, n \quad . \tag{17}$$

If the n-ring (15) is even, that is $h = +1$, all components v_i^* are periodic points of $\hat{g}_i$, and, since $\hat{g}_i$ is strictly increasing, they are fixed points of $\hat{g}_i$. The same holds true if the n-ring is odd, that is $h = -1$, and r is even. Now suppose $h = -1$ and r is odd. Then with (17) we have $\hat{g}_i^r(v_i^*(t)) = -v_i^*(t)$ and multiplication of both sides with $v_i^*(t)$ gives

$$v_i^*(t) \cdot \hat{g}_i^r(v_i^*(t) = -\,(v_i^*(t))^2 \quad , \quad i = 1, \ldots, n \quad .$$

But if v^* is not the origin, this contradicts the property

$$v_i^*(t) \cdot \hat{g}_i^r(v_i^*(t) \geq 0 \quad , \quad i = 1, \ldots, n \quad ,$$

which follows from Equation (14). Thus, for $h = -1$ the period r must be even. □

From the last argument of the proof we see, that in odd n-rings (15) there are no fixed points other than the origin, and periodic points all have an even period. There are no periodic points with period n in these networks. This can be seen as follows: Suppose v^* is an n-periodic point of Equation (15), i.e. $v_i^*(t+n) = v_i^*(t)$, $i = 1, \ldots, n$. On the other hand using Equation (16) and Lemma 1 we get

$$v_i^*(t+n) = -\,\hat{g}_i(v_i^*(t)) = -\,v_i^*(t) \quad , \quad i = 1, \ldots, n \quad .$$

Thus there is no n-periodic orbit. But then, if n is even, there are also no 2r-periodic points with $2r|n$. Especially there are no 2-periodic points in an odd n-ring with n even. But in odd n-rings there always exist 2n-periodic points, since with Lemma 1 we have

$$v_i^*(t+2n) = h^2\,(\hat{g}_i)^2(v_i^*(t)) = v_i^*(t) \quad , \quad i = 1, \ldots, n \quad .$$

Now suppose $|A \cdot S(\bar{a}(\rho))| < 1$. Then 0 is the only fixed point of $\hat{g}_i$, $i, \ldots, n$. If $|A \cdot S(\bar{a}(\rho)| > 1$, then every $\hat{g}_i$ has exactly three fixed points: 0 which is unstable, and two stable fixed points, $c_i^+ > 0$ and $c_i^- < 0$ say. Then any r-periodic point $v^* = (v_1^*, \ldots, v_n^*)$ of (15) can be represented formally by an n-tuple $\tilde{v}^*$ of the three elements $+1, 0, -1$, i.e.

$$\tilde{v}_i^* := \begin{cases} +1 & \text{if } v_i^* = c_i^+ > 0 \\ 0 & \text{if } v_i^* = 0 \\ -1 & \text{if } v_i^* = c_i^- < 0 \end{cases} \quad . \tag{18}$$

One iteration of G_ρ then corresponds to the composition $H_n \circ \pi_n$ of maps $\mathbf{R}^n \to \mathbf{R}^n$, where π_n shifts the sequence $(\tilde{v}_1^*, \ldots, \tilde{v}_n^*)$ one step cyclically to the right and the linear map H_n is given in diagonal matrix form by

$$H_n := \begin{pmatrix} h_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & h_n \end{pmatrix} \quad . \tag{19}$$

Now, if the n-ring is even, i.e. $h = +1$, the map $(H_n \circ \pi_n)^n$ is the identity map, from which follows that if a point is r-periodic, r is a devisor of n. If the n-ring is odd, i.e. $h = -1$, then $(H_n \circ \pi_n)^n$ is minus the identity map, and $(H_n \circ \pi_n)^{2n}$ is the identity map. Since there exist only periodic points with even period $2r$ in this case we have $2r|2n$ but not $2r|n$. Since the system (15) is equivalent to n-rings (3) this proves the following

Theorem 1 *For an even n-ring (3) any periodic orbit has minimum period $r \le n$ and $r|n$. For an odd n-ring any periodic orbit has minimum period $2r$, $1 \le r \le n$, with $r|n$ but not $2r|n$.*

With Lemma 1 and arguments corresponding to those given in Blum & Wang (1992) for their Corrollary 3 we also have

Lemma 2 *For $|A \cdot S(\bar{a}(\rho))| < 1$ n-rings (15) have the origin as an attractor which is the only periodic point. For $|A \cdot S(\bar{a}(\rho))| > 1$ the n-rings have exactly 3^n periodic points, one of them, the origin, is unstable, 2^n are stable, and the others are saddles.*

Using again the fact that systems (15) are equivalent to n-rings (3) we have

Theorem 2 *For $\rho \in D^+$ or $\rho \in D^-$ n-rings (3) have exactly 2^n stable periodic points.*

For $|A \cdot S(\bar{a}(\rho))| > 1$ stable periodic points v^* have components $v_i^* = c_i^+$ or $v_i^* = c_i^-$. Using the formal representations $\tilde{v}^*$ given by (18) the stable periodic points v^* of (15) are thus determined by all 2^n possible combinations of $+1$ and -1 corresponding to the 2^n corners of a hypercube in $\mathbf{R}^n$ which is centered at the origin (the unstable fixed point).

Computer simulations reveal, that the basin structure for coexisting periodic orbits is very simple. It is given by the n codimension-1 hyperplanes in v-space defined by $v_i = 0$, $i = 1, \ldots, n$, which decompose the state space $\mathbf{R}^n$ into 2^n rectangular subspaces. Each periodic point v^* lies in exactly one of these subspaces. The basin of an r-periodic attractor consists of those r subspaces which contain the r points v^* of this attractor. For the special case n=2 compare for instance (Pasemann, 1993b).

Furthermore simulations show, that for general n-rings the bifurcation sets B^+ and B^- together with the corresponding hysteresis and oscillatory domains D^+ and D^- have the same characteristic shapes as those for homogeneous n-rings which are displayed in Figure 2; they are only slightly deformed and shifted. For time varying inputs we also observe the same phenomena as described in Section 3 for homogeneous networks. If in an even n-ring the input vector $I = (I_1, \ldots, I_n)$ is allowed to move slowly along curves across the hysteresis domain D^+, only the fixed point attractors will appear, resulting in a hysteresis effect; i.e. bistability is a characteristic feature for even n-rings with time varying inputs (as long as there is no noise). Varying the inputs I of an odd n-ring across the bifurcation set B^- results in the appearance or disappearance of stable oscillations. Only one specific periodic attractor will be visible in this case.

5 Attractor structure of n-rings

To describe the dynamical behaviour of n-rings by useful output signals, we introduce the *mean activity* $\overline{o}(t)$ of the ring at time t, and the *integrated activity* ϕ_i

of unit i, where for convenience (compare Theorem 1) we integrate over a period 2n. These quantities are defined as follows:

$$\overline{o}(t) := \frac{1}{n} \sum_{i=1}^{n} o_i(t) \quad , \tag{20}$$

$$\phi_i := \frac{1}{2n} \sum_{t=1}^{2n} o_i(t) \quad . \tag{21}$$

They correspond to the spatial average of the rings activity and the temporal average of the activity of a unit.

On a periodic attractor, the mean activity $\overline{o}(t)$ of the ring will generally be periodic (compare Figures 3 and 4). Only for special ring configurations it will be constant on every periodic attractor. This typical time sequence of an attractor will be seen for example by a consecutive unit (in a different layer), which receives inputs from all units of the ring.

To characterize the different attractors we may also look at the time-sequence of activities $a_i(t_0)$, $a_i(t_0+1)$, $a_i(t_0+2), \ldots$ of unit i, $i = 1, \ldots, n$ starting at time $t = t_0$. On an r-periodic attractor these time-sequences will be periodic, i.e.

$$a_i(t_0 + t + r) = a_i(t_0 + t) \quad , \quad t = 0, 1, 2, \ldots \quad ,$$

and thus we will be interested in sequences of finite length m where, according to Theorem 1, m will be chosen to be $m = n$ or $m = 2n$ for even and odd n-rings, respectively. Thus, an *activity sequence* $\alpha_i(t, m)$ of length m of unit i is given by

$$\alpha_i(t, m) := (a_i(t), a_i(t+1), \ldots, a_i(t+m-1)) \quad .$$

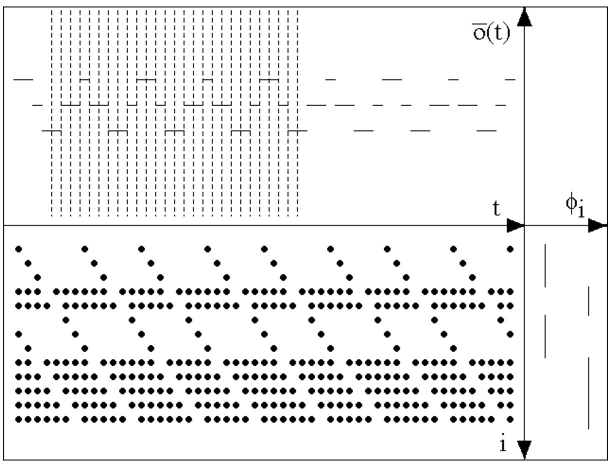

Figure 3: Firing pattern characterizing a period-13 orbit attractor in an even 13-ring (22) with four inhibitory connections: $h_1 = h_4 = h_6 = h_9 = -1$, $w = 8$.

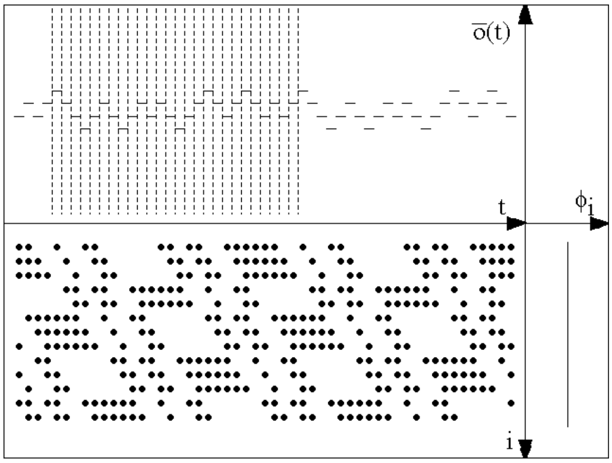

Figure 4: Firing pattern characterizing a period-26 orbit attractor in an odd 13-ring (22) with five inhibitory connections: $h_1 = h_4 = h_6 = h_9 = h_{12} = -1$, $w = 8$.

To visualize the dynamical properties on periodic attractors more clearly, we choose special n-rings (3) given by

$$w_{ii-1} = h_i \cdot w \; , \; 2\theta_i = -w_{ii-1} \; , \; w_{10} = w_{1n} \; , \; 0 < w \; , \; i = 1, \ldots, n \; , \qquad (22)$$

with $h_i \in \{+1, -1\}$ representing excitatory and inhibitory connections, respectively. In Figures 3 and 4 the activity sequences of the units of two such rings are represented by black and white dots: a black (white) dot represents $a_i > 0$ ($a_i < 0$). The corresponding mean activities $\overline{o}(t)$ and integrated activities ϕ_i are also shown.

For mathematical analysis of the general case we use the transformed coordinates v_i given by Equation (12) to introduce a *formal activity sequence* $\alpha_i(t, m)$ as a sequence of plus and minus signs, where $+$ ($-$) corresponds to $v_i(t) > 0$ ($v_i(t) < 0$). For example unit 1 in Figure 3 has the activity sequence $\alpha_1(t, 13) = (+, -, -, -, -, -, -, +, -, -, -, -, -)$. In the following we will consider only formal activity sequences $\alpha_i(t, m)$. Attractor dynamics of n-rings may then be discussed in terms of symbolic dynamics as for example in Lewis & Glass (1992). Furthermore we will give the following definitions:

Two sequences $\alpha(t, m)$, $\beta(t, m)$ are *equivalent* iff there exists a $1 \leq q < m$ such that $\alpha(t, m) = \beta(t + q, m)$. Since an equivalence class does not depend on t we denote it by $\alpha[m]$. An equivalence class $\alpha[m]$ is called *r-periodic*, iff it is composed of q equivalence classes $\beta[r]$ with $r|m$ and $q \cdot r = m$, i.e. $\alpha[m] = (\beta[r], \ldots, \beta[r])$.

An equivalence class $\alpha^d[m]$ is called *dual* to $\alpha[m]$ iff the plus signs are replaced by minus signs and vice versa. For representatives this means for example that the sequence $(+, -, +, +)$ is dual to $(-, +, -, -)$ and also to $(+, -, -, -)$. A class $\alpha[m]$ is called *self-dual* iff for representatives there exists a $1 \leq q < m$ such that

$\alpha^d(t,m) = \alpha(t+q,m)$. For instance the equivalence class $(+,+,+,-,-,-)$ is self-dual with $q=3$. Self-dual sequences must have even length, i.e. $m=2r$.

To describe the effect of an inhibitory connection in a ring we define the concept of a group of units on the ring as follows: A *group* $\Gamma_i(q)$ consists of $q+1$ consecutive units on the ring, such that its first unit i receives an inhibitory input, i.e. $h_i = -1$, and the following q units receive an excitatory input, i.e. q is the largest number such that $h_{i+1} = \ldots = h_{i+q} = +1$ is satisfied.

The activity sequences of consecutive units belonging to a group $\Gamma_i(q)$ are shifted by one time step, and we have

$$\alpha_i(t,m) = \alpha_{i+s}(t+s,m) \quad , \quad 1 \le s \le q \quad .$$

Each group has its characteristic "firing pattern" as can be seen for example in Figures 3 and 4. The *firing pattern* of the ring (group) is defined as the set of formal activity sequences of the units of the n-ring (group). It corresponds to a representation of a specific attractor of the n-ring. The groups will remain the same on all attractors, since, by definition, they reflect the distribution of inhibitory connections of the ring.

If s denotes the number of inhibitory connections in an n-ring then for $s=0$ (a purely excitatory ring) there exists exactly one group, the whole ring, and for $1 \le s \le n$ there are s groups on the ring. Thus, for a purely inhibitory ring every unit is a group.

Lemma 3 *In an n-ring network with $1 \le s \le n$ inhibitory connections the units are arranged into s different groups. On any attractor consecutive groups on the ring have dual formal activity sequences.*

Proof: Let $\Gamma_i(q)$ and $\Gamma_j(q')$ denote two consecutive groups on a ring, i.e. $j = i+q+1$, $h_j = -1$. Using formal activity sequences and denoting the formal activity of unit i at time t by $\tilde{v}_i(t) \in \{+1,-1\}$ the dynamics on an attractor is described by the map $(\pi_n \circ H_n)$ defined in the last section (Equation (19)). For the components, iterating one step thus gives $\tilde{v}_i(t+1) = h_i \tilde{v}_{i-1}(t)$ and we have

$$\tilde{v}_j(t) = h_j \tilde{v}_{i+q}(t-1) = h_j h_{i+q} \cdots h_{i-1} \tilde{v}_i(t-q-1) \quad , \quad \text{for all } t \, .$$

But since $h_{i+q} = \cdots = h_{i-1} = +1$ by definition of a group, we have

$$\tilde{v}_j(t) = -\tilde{v}_i(t-q-1) \quad , \quad \text{for all } t \, .$$

Thus for the formal activity sequences we have

$$\alpha_j^*(t,m) = (\alpha_i^*)^d(t-q-1,m) \quad ,$$

and the firing patterns of the groups Γ_i and Γ_j are generated by dual equivalence classes. □

Theorem 3 *Every attractor of an n-ring (even or odd) is characterized by exactly one equivalence class $\alpha[n]$ of formal activity sequences of length n. For even n-rings firing patterns on any attractor are generated by a class $\alpha[n]$ and its dual $\alpha^d[n]$ if there are s inhibitory connections with $s \geq 2$. For odd n-rings firing patterns on an attractor are generated by a self-dual class $\beta[2n]$ which is composed of $\alpha[n]$ and $\alpha^d[n]$.*

Proof: According to Theorem 1 firing patterns on periodic attractors for even n-rings are characterized by activity sequences of length n. If there is only one group in the ring, then on each attractor they are generated by exactly one class $\alpha[n]$. Lemma 3 shows that if there exist more then one group on the ring, then each firing pattern is generated by a class $\alpha[n]$ and its dual.

For odd n-rings Theorem 1 says, that firing patterns on periodic attractors are characterized by activity sequences of length 2n. That they are self-dual, i.e.

$$\beta_i(t, 2n) = (\alpha_i(t, n), \alpha_i^d(t + n, n)) \quad .$$

follows from Equation (16) which for formal activities simply reads $\tilde{v}_i(t + n) = -\tilde{v}_i(t)$. Therefore on each attractor the activity sequences, and thus the whole firing pattern, are generated again by one class $\alpha[n]$ and its dual. □

From Theorem 3 it follows that the number of coexistent attractors in an n-ring corresponds to the number of possible equivalence classes $\alpha[n]$ satisfying the requirements of Theorem 1. In particular, it does not depend on the number of inhibitory connections. Therefore it can be calculated in a purely algebraic way. But first we give

Example 1 Consider 3-rings (3) with $\rho \in D^+$ or D^-. Coexistent attractors in even 3-rings are characterized by the four equivalence classes

$$\begin{array}{llll} \alpha^1[3] = (+,+,+) & , \quad \alpha^2[3] = (-,-,-) & : & \text{fixed point attractors,} \\ \alpha^3[3] = (+,-,-) & , \quad \alpha^4[3] = (+,+,-) & : & \text{period-3 attractors.} \end{array}$$

So there will be two fixed points and two period-3 attractors, all coexistent. For odd 3-rings there exist two such classes given by

$$\beta^1[6] = (+,+,+,-,-,-) \quad : \quad \text{period-6 attractor,}$$

$$\beta^2[6] = (+,-,+,-,+,-) \quad : \quad \text{period-2 attractor.}$$

The relative placement of these activity sequences depends of course on the distribution of the inhibitory connections in the ring. This will result in different firing patterns for rings with different distributions of inhibitory connections, and thus in different mean activities $\overline{o}(t)$ and integrated activities ϕ_i.

Theorem 4 *The number $\Lambda(r)$ of r-periodic equivalence classes $\alpha[m]$, $1 \leq r \leq m$ with $r|m$, is given recursively by*

$$\begin{aligned} \Lambda(1) &= 2 \quad , \\ \Lambda(r) &= \frac{1}{r}\,(2^r - \sum_{k=1}^{s} \frac{r}{q_k}\,\Lambda(\frac{r}{q_k})) \quad , \end{aligned} \tag{23}$$

where q_k, $k = 1, \ldots, s$ denote all divisors of r satisfying $1 \leq q_k < r$. The number $\Delta(2r)$ of 2r-periodic self-dual equivalence classes $\beta[2m]$ with $1 \leq r \leq m$, $r|m$ but not $2r|m$ is given recursively by

$$\begin{aligned} \Delta(2) &= 1 \quad , \\ \Delta(2r) &= \frac{1}{2r}\,(2^r - \sum_{k=1}^{s} \frac{2r}{q_k}\,\Delta(\frac{2r}{q_k})) \quad , \end{aligned} \tag{24}$$

where q_k with $1 < q_k \leq n$, $k = 1, \ldots, s$ denote all odd *numbers which are devisors of r.*

Proof: Consider r-periodic equivalence classes $\alpha[m]$. They are composed of $q = \frac{m}{r}$ equivalence classes $\alpha'[r]$. For a given r, there are 2^r different activity sequences of length r. For $r = 1$ there are two equivalence classes, given by $(+)$ and $(-)$. So $\Lambda(1) = 2$. Suppose now $r > 1$ and $1 \leq q_k < r$, $k = 1, \ldots, s$ denote all devisors of r. Then among the possible 2^r activity sequences of length r there are $\frac{r}{q_k}\,\Lambda(\frac{r}{q_k}))$ activity sequences, which are $\frac{r}{q_k}$-periodic, $k = 1, \ldots, s$. Thus the number $\Lambda(r)$ of r-periodic equivalence classes $\alpha[m]$, $1 \leq r \leq m$, is given by the recurrent formula (23) of the Theorem.

Consider now self-dual equivalence classes $\beta[2m]$ which are periodic. They must be composed of $q = \frac{2m}{2r}$ self-dual equivalence classes $\beta'[2r] = (\alpha[r], \alpha^d[r])$. But since $\beta[2m]$ is build from q copies of a self-dual equivalence class $\beta'[2r]$, $2r$ can not be a devisor of m, and thus q must be an odd number. For $r = 1$ there exist only one self-dual equivalence class of length $2r = 2$ represented by $\beta[2] = (+, -)$. So $\Delta(2) = 1$. Repeating the arguments of the first part of the proof, now for self-dual equivalence classes, we end up with the recurrent formula (24). □

Corollary 1 *If an n-ring (3) is even with $\rho \in D^+$, the number of its r-periodic attractors, $1 \leq r \leq n$, $r|n$ is given by $\Lambda(r)$, and the total number $E(n)$ of its coexistent periodic attractors by*

$$E(n) = \sum_{k=1}^{s} \Lambda(r_k) \quad , \quad 1 \leq r_k \leq n \quad , \quad r_k|n \quad .$$

If an n-ring (3) is odd with $\rho \in D^-$, the number $\Delta(2r)$ of its 2r-periodic attractors, with $r|n$ but not $2r|n$, $1 \leq r \leq n$ is given by $\Delta(2r)$, and the total number $O(n)$

of its coexistent periodic attractors by

$$O(n) = \sum_{k=1}^{s} \Delta(2r_k) \quad , \quad 1 < r_k \leq n \quad , \quad r_k|n, \text{ but not } 2r_k|n.$$

Proof: Since there is a one-to-one correspondence between periodic attractors of an n-ring and equivalence classes of formal activity sequences, the number of r-periodic attractors is given by Equation (23) and Equation (24) of Theorem 3, respectively. To get the total number of periodic attractors we just have to add the numbers $\Lambda(r)$ and $\Delta(r)$, respectively, for those periods r, which can appear in the n-ring under consideration according to Theorem 1. □

Example 2 Denoting the attractor structure for an even n-ring network by $(n)^{\Lambda(n)} + (r_1)^{\Lambda(r_1)} + \cdots + (r_s)^{\Lambda(r_s)}$, where $n, r_1, \ldots, r_s$ are the appearing periods, Theorem 1 and Corollary 1 will give for example (compare also Appendix 1)

$$\begin{array}{llll} n = 13: & (13)^{630} + (1)^2 , & E(13) = 632 & , \\ n = 14: & (14)^{1161} + (7)^{18} + (2)^1 + (1)^2 , & E(14) = 1182 & , \\ n = 15: & (15)^{2182} + (5)^6 + (3)^2 + (1)^2 , & E(15) = 2192 & , \\ n = 16: & (16)^{4080} + (8)^{30} + (4)^3 + (2)^1 + (1)^2 , & E(16) = 4116 & . \end{array}$$

Denoting the attractor structure for an odd n-ring network correspondingly by $(2n)^{\Delta(2n)} + (2r_1)^{\Delta(2r_1)} + \cdots + (2r_s)^{\Delta(2r_s)}$ we have

$$\begin{array}{llll} n = 13: & (26)^{315} + (2)^1 , & O(13) = 316 & , \\ n = 14: & (28)^{585} + (4)^1 , & O(14) = 586 & , \\ n = 15: & (30)^{1091} + (10)^3 + (6)^1 + (2)^1 , & O(15) = 1096 & , \\ n = 16: & (32)^{2048} , & O(16) = 2048 & . \end{array}$$

Different groups will of course not be discernible by the mean activity $\overline{o}(t)$ of the n-ring, but for special n-rings (22) like the one underlying Figure 3, the groups are in general distinguishable by the characteristic integrated activity ϕ_i of their units. This is not the case, iff the attractor is represented by self-dual activity sequences. In the case of general n-rings this discrimination will not be so clearly seen.

6 Summary and Discussion

We may first remark that the results obtained for the discrete dynamics of n-rings are different from those for the continuous case. The standard continuous dynamics of an n-ring is given by

$$\dot{a}_i = -\gamma_i a_i + \theta_i + w_{ii-1}\sigma(a_{i-1}) \quad , \quad w_{10} = w_{1n} \quad , \quad i = 1, \ldots, n \quad , \qquad (25)$$

where $\gamma_i > 0$ denotes the constant decay rate of unit i. Observe, that Equation (3) can be read as a discretized version of Equation (25) given by

$$a_i(t + \Delta t) := (1 - \Delta t)\gamma_i a_i(t) + \Delta t(\theta_i + w_{ii-1}\sigma(a_{i-1}(t)) \quad i = 1, \ldots, n \quad ,$$

with $\gamma_i = 1$ and $\Delta t = 1$. However, for the continuous dynamics (25) here are some known analytical results:

- A continuous even n-ring can not exhibit stable oscillatory behaviour. This is due to (Hirsch, 1989), where it is shown, that an irreducible network with the even loop property (i.e. such that the number of inhibitory connections along any directed loop in the network is even) has almost quasiconvergent dynamics.

This is quite different from the discrete dynamics where stable oscillations can occur according to Theorem 1.

- Continuous odd n-rings can have stable periodic behaviour, and Hopf bifurcations are observed.

The existence of Hopf bifurcations for odd rings was demonstrated e.g. in Atiya & Baldi (1989). In an der Heiden (1980) oscillations in continuous odd neural ring networks was analysed, proving that stable oscillatory behaviour exists whenever the steady state is unstable. Moreover the attractor structure was described already by formal activity sequences. Although the discussed system was slightly more general then the system (25), the firing pattern for a 3-ring, given there as an example, corresponds to that of a discrete 3-ring generated by $\beta_1(6)$ in Example 1 above.

At first sight, the concept of a group of units on the ring, introduced in Section 5, resembles that of a cell-assembly introduced by Hebb (Hebb, 1949). A cell-assembly is considered as a set of units in a network firing coherently, or jointly with an increased rate, in response to a given input signal (Palm, 1982). This means, that a unit may belong to different assemblies, since an assembly refers to a specific input (or a class of inputs). A group, in contrast, reflects the distribution of inhibitory connections in the ring. The definition of a group is thus independent from specific input configurations. A unit of the ring always belongs to one and the same group.

Including the results for a single unit with self-connection (Pasemann, 1993a), the results on n-rings prove that discrete time one-loop networks can exhibit only convergent or oscillatory behaviour. To show more complex behaviour, i.e. displaying quasiperiodic or chaotic discrete dynamics, neural networks with nonlinear additive units must have at least two closed loops, where one loop can be realized as a self-connection. This was observed for example in Chapeau-Blondeau & Chauvet (1992).

Using n-rings as subnetworks (modules) in a larger network, they can provide hysteresis effects and controlled oscillations, respectively. As discussed already e.g. in Harth, Csermely, Beek, & Lindsay (1970), Wilson & Cowan (1972), hysteresis effects may be useful for short term memory function. But since the same effect appears already for a single unit with excitatory self-connection (Pasemann, 1993a), it might not be effective to use an n-ring for this purpose. On the other hand, their large number of coexisting stable periodic orbits suggests, that they can be used as modules for temporal coding of information. A method for storing and retrieving information based on stable periodic orbits of one-dimensional maps was discussed e.g. in Andreyev et. al. (1993). Storage of information in stable limit cycles of neural networks was considered e.g. in Baird (1986), and Li & Hopfield (1989).

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Appendix 1: (not published in *Neural Networks*)

In the following we present the attractor structure for n-ring networks for $n = 1, \ldots, 20$ according to Theorem 1 and Corollary 1. The case $n = 1$ corresponds to a single unit with self-interaction as was discussed in (Pasemann, 1993a). Denoting the structure for an *even* n-ring network by $(n)^{\Lambda(n)} + (r_1)^{\Lambda(r_1)} + \cdots + (r_s)^{\Lambda(r_s)}$, where $n, r_1, \ldots, r_s$ are the appearing periods, and the total number of attractors by $E(n)$ we get the following results:

$$
\begin{array}{rcl}
n = 1: & (1)^2 \,, & E(1) = 2 \,, \\
n = 2: & (2)^1 + (1)^2 \,, & E(2) = 3 \,, \\
n = 3: & (3)^2 + (1)^2 \,, & E(3) = 4 \,, \\
n = 4: & (4)^3 + (2)^1 + (1)^2 \,, & E(4) = 6 \,, \\
n = 5: & (5)^6 + (1)^2 \,, & E(5) = 8 \,, \\
n = 6: & (6)^9 + (3)^2 + (2)^1 + (1)^2 \,, & E(6) = 14 \,, \\
n = 7: & (7)^{18} + (1)^2 \,, & E(7) = 20 \,, \\
n = 8: & (8)^{30} + (4)^3 + (2)^1 + (1)^2 \,, & E(8) = 36 \,, \\
n = 9: & (9)^{56} + (3)^2 + (1)^2 \,, & E(9) = 60 \,, \\
n = 10: & (10)^{99} + (5)^6 + (2)^1 + (1)^2 \,, & E(10) = 108 \,, \\
n = 11: & (11)^{186} + (1)^2 \,, & E(11) = 188 \,, \\
n = 12: & (12)^{335} + (6)^9 + (4)^3 + (3)^2 + (2)^1 + (1)^2 \,, & E(12) = 352 \,, \\
n = 13: & (13)^{630} + (1)^2 \,, & E(13) = 632 \quad , \\
n = 14: & (14)^{1161} + (7)^{18} + (2)^1 + (1)^2 \,, & E(14) = 1182 \quad , \\
n = 15: & (15)^{2182} + (5)^6 + (3)^2 + (1)^2 \,, & E(15) = 2192 \quad , \\
n = 16: & (16)^{4080} + (8)^{30} + (4)^3 + (2)^1 + (1)^2 \,, & E(16) = 4116 \quad . \\
n = 17: & (17)^{7710} + (1)^2 \,, & E(17) = 7712 \,, \\
n = 18: & (18)^{14532} + (9)^5 6 + (6)^9 + (3)^2 + (2)^1 + (1)^2 \,, & E(18) = 14602 \,, \\
n = 19: & (19)^{27594} + (1)^2 \,, & E(19) = 27596 \,, \\
n = 20: & (20)^{52377} + (10)^9 9 + (5)^6 + (4)^3 + (2)^1 + (1)^2 \,, & E(20) = 52488 \,.
\end{array}
$$

Correspondingly, denoting the attractor structure for an *odd* n-ring network by $(2n)^{\Delta(2n)} + (2r_1)^{\Delta(2r_1)} + \cdots + (2r_s)^{\Delta(2r_s)}$ and the total number of attractors by $O(n)$ we have

$$
\begin{array}{rcl}
n=1: & (2)^1\ , & O(1)=1\ , \\
n=2: & (4)^1\ , & O(2)=1\ , \\
n=3: & (6)^1+(2)^1\ , & O(3)=2\ , \\
n=4: & (8)^2\ , & O(4)=2\ , \\
n=5: & (10)^3+(2)^1\ , & O(5)=4\ , \\
n=6: & (12)^5+(4)^1\ , & O(6)=6\ , \\
n=7: & (14)^9+(2)^1\ , & O(7)=10\ , \\
n=8: & (16)^{16}\ , & O(8)=16\ , \\
n=9: & (18)^{28}+(6)^1+(2)^1 30\ , & O(9)=30\ , \\
n=10: & (20)^{51}+(4)^1\ , & O(10)=52\ , \\
n=11: & (22)^{93}+(2)^1\ , & O(11)=94\ , \\
n=12: & (24)^{170}+(8)^2\ , & O(12)=172\ , \\
n=13: & (26)^{315}+(2)^1\ , & O(13)=316\ , \\
n=14: & (28)^{585}+(4)^1\ , & O(14)=586\ , \\
n=15: & (30)^{1091}+(10)^3+(6)^1+(2)^1\ , & O(15)=1096\ , \\
n=16: & (32)^{2048}\ , & O(16)=2048\ , \\
n=17: & (34)^{3855}+(2)^1\ , & O(17)=3876\ , \\
n=18: & (36)^{7280}+(12)^5+(4)^1\ , & O(18)=7286\ , \\
n=19: & (38)^{13797}+(2)^1\ , & O(19)=13798\ , \\
n=20: & (40)^{26214}+(8)^2\ , & O(20)=26216\ .
\end{array}
$$