

A Simple Chaotic Neuron ^{*}

Frank Pasemann

Max-Planck-Institute for Mathematics in the Sciences

Inselstr. 22-26, D-04103 Leipzig, Germany

email: f.pasemann@mis.mpg.de

Abstract

The discrete dynamics of a dissipative nonlinear model neuron with self-interaction is discussed. For units with self-excitatory connection hysteresis effects, i.e. bistability over certain parameter domains, are observed. Numerical simulations demonstrate that self-inhibitory units with non-zero decay rates exhibit complex dynamics including period doubling routes to chaos. These units may be used as basic elements for networks with higher-order information processing capabilities.

^{*}appeared in: *Physica D*, **104**, 205 - 211, 1997.

1 Introduction

Biological neurons exhibit a large variety of dynamical behaviors even when they are not embedded in a network. This type of dynamics is captured by biologically inspired neuron models like the Hodgkin-Huxley [1] or the FitzHugh-Nagumo equations [2], [3]. On the other hand, formal neurons used in artificial neural networks like the McCulloch-Pitts neuron or the graded response neurons [4] have only trivial, i.e. convergent dynamics as single elements.

With the aim to model also higher-order brain-like functions and to account for collective or clustering effects observed in biological brains (e.g. synchronized oscillations, see [5]) alternative elements with more complex behaviors have been taken into account. These were for instance binary elements with dynamical thresholds [6], formal spiking neurons (e.g. [7]), phase oscillators (e.g. [8]), dynamical perceptrons [9], or just the logistic map [10], which has no neurobiological motivation at all. A biologically motivated formal chaotic neuron was introduced in [11], and its role as basic element for higher-order information processing networks was discussed (see also [12] and the articles cited there).

In [13] the discrete dynamics of a standard graded response neuron with self-interaction was analyzed, indicating that it can oscillate with period two (if self-inhibitory) or can show hysteresis effects (if self-excitatory). Adding a non-zero decay term to this neuron model already provides complex single neuron dynamics including period doubling routes to chaos. This is discussed in the following section where analytical and numerical results are presented. A short discussion follows in Sec. 3.

2 Single neuron dynamics

In the following we will discuss the discrete activation dynamics of a dissipative additive nonlinear neuron corresponding to the one dimensional map $f : \mathbf{R} \rightarrow \mathbf{R}$ given by

$$a(t+1) = \gamma \cdot a(t) + \theta + w \cdot \sigma(a(t)), \quad 0 \leq \gamma < 1, \quad (1)$$

with bias term θ , self-weight w , dissipation γ , and neuron output given by the sigmoidal transfer function $\sigma(x) := (1 + e^{-x})^{-1}$. The parameter θ will be considered as the sum of a fixed bias term θ_0 and the external input I of the neuron. The decay rate γ reflects the dissipative properties of real neurons.

For $\gamma = 0$ the dynamics (1) was analysed in [13] revealing three different domains in the (θ, w) -parameter space (compare Fig. 1). For a self-excitatory neuron, i.e. $w > 0$, there exists a hysteresis domain II over which the system has two coexisting fixed point attractors. This domain is bounded by a bifurcation set, which is determined by a cusp catastrophe at $(\theta_c, w_c) = (-2, 4)$. For self-inhibitory neurons, i.e. $w < 0$, there is an oscillatory domain III corresponding to global period-2 orbit attractors. It starts at the critical point $(\theta_c, w_c) = (2, -4)$.

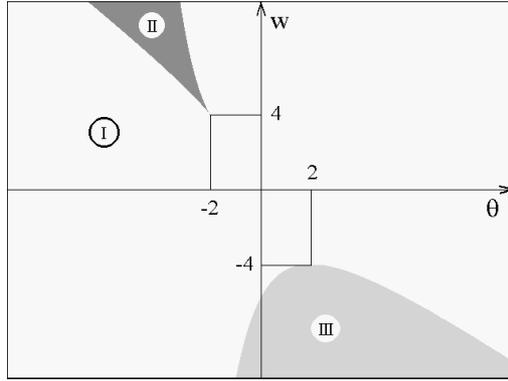


Figure 1: (θ, w) -parameter domains for $\gamma = 0$ indicating global fixed point attractors (I), bistability (hysteresis) (II), and global period-2 orbit attractors (III).

For parameter values outside these two domains the dynamics (1) has only global fixed point attractors (domain I).

Analogous simulations for $\gamma > 0$ show, that the shape of the bifurcation sets B^+ (fold bifurcations at the boundary of the hysteresis domain) and B^- (period doubling bifurcations at the boundary of the oscillatory domain) stays qualitatively the same. The domains are slightly deformed and the critical points are shifted with varying γ as calculated analytically below.

The stability condition for fixed points a^* satisfying

$$a^* = \gamma \cdot a^* + \theta + w \cdot \sigma(a^*), \quad 0 \leq \gamma < 1$$

is given by the equation

$$|\gamma + w \cdot \sigma'(a^*)| < 1. \quad (2)$$

Since $0 < \sigma'(x) \leq 1/4$ for all x , all stationary states will be stable for $|4 \cdot \gamma + w| < 4$. For $-2 \cdot \theta = w$ the origin $a^* = 0$ is a fixed point of the dynamics for all γ . Thus, for $w = 4(1 - \gamma)$ and $w = -4(1 + \gamma)$ the fixed point $a^* = 0$ is nonhyperbolic with eigenvalue $+1$ and -1 , respectively.

Using the fixed point equation together with the stability condition (2), and choosing as parameter $\tau \cong a^*$, one can describe the bifurcation sets B^+ and B^- analytically as parameterized curves $\mathbf{R} \rightarrow \mathbf{R}^2$ in (θ, w) -parameter space as follows:

$$B^+ : \quad \theta(\tau) = (1 - \gamma)\tau - \frac{1 - \gamma}{1 - \sigma(\tau)}, \quad w(\tau) = \frac{1 - \gamma}{\sigma'(\tau)}, \quad \tau \in \mathbf{R}, \quad (3)$$

where the critical point (w minimal) is given by $\tau = 0$, and

$$B^- : \quad \theta(\tau) = (1 - \gamma)\tau + \frac{1 + \gamma}{1 - \sigma(\tau)}, \quad w(\tau) = -\frac{1 + \gamma}{\sigma'(\tau)}, \quad \tau \in \mathbf{R}, \quad (4)$$

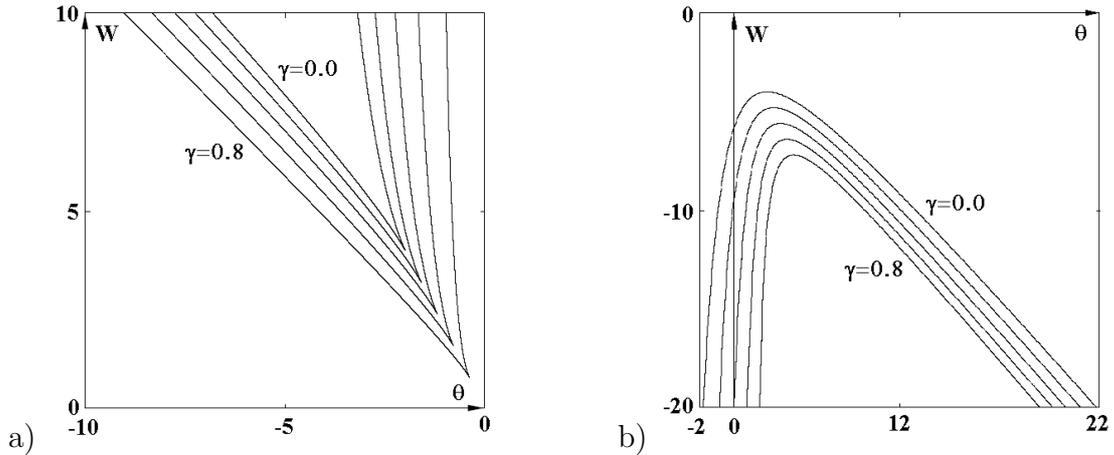


Figure 2: Bifurcation sets in (θ, w) -parameter space for $\gamma = 0.0, 0.2, 0.4, 0.6,$ and 0.8 : a.) B^+ indicating saddle-node bifurcations, and b.) B^- indicating period-doubling bifurcations.

with critical point (w maximal) given again by $\tau = 0$.

Figures 2a,b reveal, that for fixed γ and w bifurcation points on B^+ and B^- are always symmetric around the line $\theta = -\frac{1}{2}w$. This corresponds to a general property of the map f . Introducing the new parameter $\xi := \theta + \frac{1}{2}w$, and rewriting f now as

$$f_{\xi}(a) := \gamma a + \xi + w \cdot (\sigma(a) - 0.5),$$

we see, using the symmetry $\sigma(-a) = 1 - \sigma(a)$, that $f_{\xi}(-a) = -f_{-\xi}(a)$. So we have the property: if a^* is a periodic point for f_{ξ} then $-a^*$ will be a periodic point for $f_{-\xi}$; i.e., in general, with γ and w fixed, the system will have the same qualitative dynamics for ξ and $-\xi$.

2.1 Self-excitatory neuron

We now consider self-excitatory neurons; i.e. $w > 0$. We observe that the map f giving the dynamics (1) is monotonically increasing for all γ . Following the line of analysis as in [13], we can show that again there is a hysteresis domain defined by a cusp catastrophe potential. The critical point (cusp catastrophe) in (θ, w) -parameter space now depends on γ , and is given by

$$(\theta_c, w_c)(\gamma) = (-2 \cdot (1 - \gamma), 4 \cdot (1 - \gamma)).$$

For fixed γ the hysteresis domain is enclosed by the bifurcation set B^+ given by (3) and displayed in Fig. 2a. For (θ, w) values in this domain one can show that the system has one unstable and two stable fixed points. Outside this domain there exists a global fixed point attractor. Varying θ for fixed $w > w_c$ back and forth over this domain will result in a hysteresis effect. For $\gamma > 0$ the dynamics

is qualitative not different from the case $\gamma = 0$ studied in [13]. Thus also this type of neuron may serve as a short-term memory element.

2.2 Self-inhibitory neuron

Contrary to the self-excitatory case the self-inhibitory neuron ($w < 0$) reveals a more complex behavior for $\gamma > 0$ than for $\gamma = 0$. But again we observe bifurcations from global fixed point attractors to global period-2 orbit attractors at the bifurcation set B^- given by equation (4) and displayed in Fig. 2b. For fixed γ the critical point is given by

$$(\theta_c, w_c)(\gamma) = (2 \cdot (1 - \gamma), -4 \cdot (1 - \gamma)).$$

Inside the oscillatory (θ, w) -domain enclosed by B^- there exist now two chaotic regions which are reached via period doubling routes to chaos, as can be seen in Fig. 3a showing oscillatory (shaded) and chaotic (black) domains in (θ, w) -parameter subspace for $\gamma = 0.8$. They are separated by the line $-2\theta = w$, and, with γ and w fixed, θ values for bifurcations lie symmetric around $\theta = -\frac{1}{2}w$ as is suggested by the properties of the map f_ξ described above. This can be more clearly seen in Fig. 3b displaying oscillatory and chaotic domains in (θ, γ) -subspace for $w = -16$. A corresponding bifurcation diagram for $0 \leq \theta \leq 8 = -\frac{1}{2} \cdot w$, with $\gamma = 0.6$ and $w = -16$ fixed, is displayed in Fig. 4 indicating the period doubling routes to chaos. The same route to chaos is observed in the bifurcation diagram (Fig. 5) for $0 < \gamma < 1$ with $\theta = 4$ and $w = -16$ fixed.

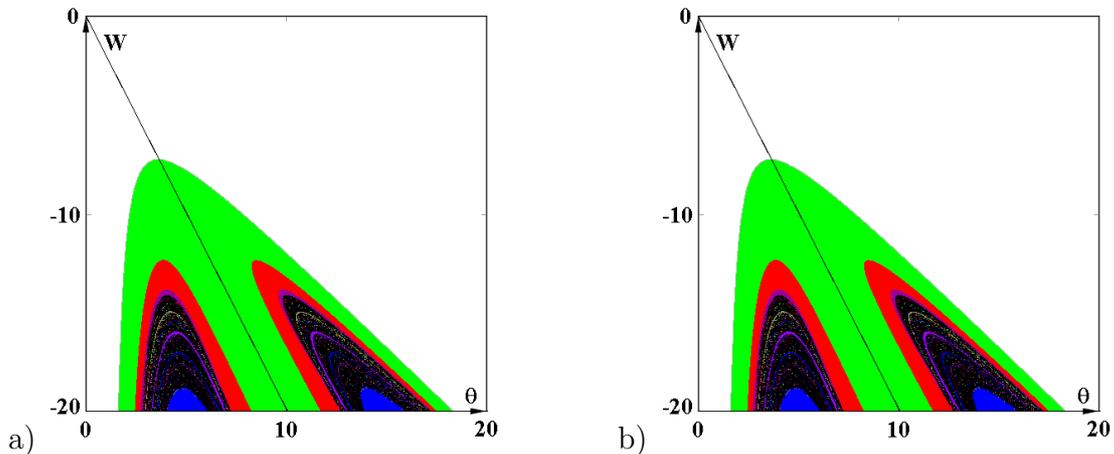


Figure 3: a.) (θ, w) -parameter domains ($\gamma = 0.8$) and b.) (θ, γ) -parameter domains ($w = -16$) for periodic and chaotic attractors indicating period-doubling routes to chaos. White: fixed point attractors, black: chaos and periods above 4.

In fact, chaotic dynamics has to be expected for $\gamma > 0$ because in this case the map f defining the dynamics (1) is a bimodal map satisfying $f' > 1$ inside

the domain enclosed by the bifurcation set B^- . The two extrema will appear for a satisfying $\gamma = -w\sigma'(a)$, and, since σ' is a symmetric function, we will have two of them at a and $-a$. For $\gamma = 0$ this condition can be satisfied only for $a \rightarrow \pm\infty$, but for $\gamma > 0$ finite a will do. Chaos is also indicated by period-3 domains appearing for example in Fig. 3a around $(\theta, w) = (5, -20)$ and, correspondingly, $(\theta, w) = (15, -20)$ with $\gamma = 0.6$.

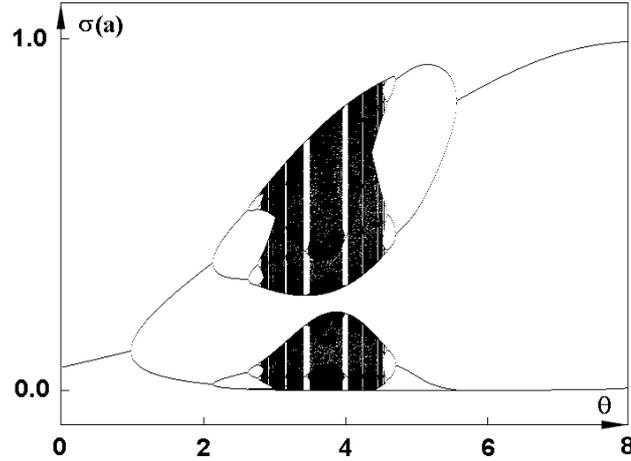


Figure 4: Bifurcation diagram for θ with $\gamma = 0.6$ and $w = -16$ fixed, indicating a period doubling route to chaos.

We now want to show that the period-2 orbits existing along the separating line $-2\theta = w$ for $w < w_c < 4 \cdot (1 - \gamma)$ and $\gamma > 0$ are always stable (compare Fig. 3a). For this set of parameters we define the function f_0 , corresponding to f_ξ for $\xi = 0$, by

$$f_0(a) := \gamma a + w(\sigma(a) - 0.5),$$

which is antisymmetric, i.e. $f_0(-a) = -f_0(a)$. Thus the period-2 orbits will have the form $\{a^*, -a^*\}$. The stability condition for these orbits reads

$$|(f_0 \circ f_0)'(a^*)| = |f_0'(a^*) \cdot f_0'(-a^*)| < 1.$$

which gives, since σ' is symmetric,

$$|(\gamma + w\sigma'(a^*)) \cdot (\gamma + w\sigma'(-a^*))| = |(\gamma + w\sigma'(a^*))|^2 < 1.$$

With $w < -4(1 - \gamma)$, $0 < \sigma'(a^*) < \frac{1}{4}$, and $0 < \gamma < 1$ this condition is always satisfied.

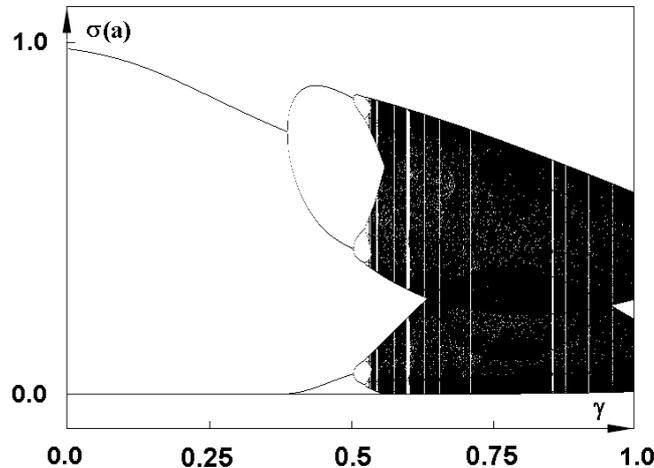


Figure 5: Bifurcation diagram for γ with $\theta = 4$ and $w = -16$ fixed, again indicating a period doubling route to chaos.

3 Discussion

We analysed the dynamics of a single dissipative formal neuron and compared it with the dynamics of the standard neuron model ($\gamma = 0$). Concerning self-excitatory case, we have seen that adding a non-zero decay term to the standard neuron model will not alter the qualitative behavior of this unit. For larger self-excitation ($w > 4(1-\gamma)$) it will continue to have a parameter domain of bistability, so that hysteresis is still a typical dynamical effect also of the dissipative self-excitatory element. These units may serve as short-term memory building blocks.

A different statement holds for self-inhibitory neurons. Adding a non-zero decay term here results in the appearance of more complex behavior, characterized by the existence of chaotic dynamics. This was already revealed in a similar model by Aihara *et.al.* [11] where an additional parameter ϵ controlling the steepness of the logistic function was introduced (here we have set $\epsilon = 1$). They also discussed the biological relevance of this model (see also [12]).

We shall remark that the results obtained here do not depend on the specific choice of the sigmoide. The existence of chaos for instance is based on the fact, that with a decay rate $\gamma > 0$ the self-inhibitory neuron dynamics is given by a bimodal map, which is true for all sigmoides and convenient parameter values.

Equation (1) can also be read as a discretized version of the often used continuous neuron dynamics [14]

$$\frac{d}{dt}a(t) = -\gamma' \cdot a(t) + \theta + w \cdot \sigma(a(t)), \quad 0 < \gamma' \leq 1. \quad (5)$$

With a discretization given by

$$a(t + \Delta t) = (1 - \gamma' \Delta t)a(t) + \Delta t(\theta + w\sigma(a))$$

equation (1) is obtained for any Δt and corresponding rescaled parameters γ' , θ , and w . But this type of discretization does not make sense in general, since a continuous one-dimensional system does not display chaotic dynamics. Although for the self-excitatory case of the continuous dynamics (5) one observes also hysteresis effects (compare [13]), the dissipative self-inhibitory neuron is convergent, i.e., it exhibits only fixed point attractors. Instead, one may consider (1) as a discretization of a one-dimensional differential equation $da(t)/dt = f(a(t)) + g(a(t - \tau))$ with time delay τ . These systems can have chaotic dynamics [15]. So starting from the continuous system

$$\frac{d}{dt}a(t) = -a(t) + \theta + \gamma a(t - \tau) + w\sigma(a(t - \tau)), \quad (6)$$

and using the change of variable $t' = t/\tau$, we obtain the discrete version (1) by making τ tend to infinity, i.e. making the delay long enough for the system to relax to a fixed point, or, equivalently fix $\tau = 1$ and letting the relaxation time go to zero. The biological relevance of equation (6) is of course not immediately obvious.

The importance of chaotic dynamics for higher-order information processing (brain) functions was stressed by several authors in recent years; see for instance [16], [17], [18], [8], [11], and [10]. The simple dissipative self-inhibitory neuron considered here may serve as a basic element for artificial neural networks with extended computational capabilities. Using for instance a coupled map lattice approach [10], networks of these chaotic elements will display specific complex spatio-temporal effects like synchronization of subsystems or clustering of neurons by phase, amplitude and/or frequency. More sophisticated architectures, using for instance a modular structure, which utilize these complex neurodynamical properties may be able to provide a new kind of information processing.

References

- [1] A. L. Hodgkin, A. F. Huxley, A quantitative description of membrane current and its application to conduction and excitation in nerve, *J. Physiol.*, **117**, 500-544, 1952.
- [2] R. FitzHugh, Mathematical models of excitation and propagation in nerve, in: H. P. Schwan (ed.), *Biological Engineering*, McGraw-Hill, New York, 1969, pp. 1-85.
- [3] J. Nagumo, S. Arimoto, and S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proc. IRE*, **50**, 2061-2070, 1962.
- [4] J. A. Hertz, A. Krogh, and R. G. Palmer, *Introduction to the Theory of Neural Computation*, Addison-Wesley, Redwood City, 1991.
- [5] W. Singer, Time as coding space in neocortical processing, in: G. Buzsáki, R. Llinás, W. Singer, A. Berthoz, and Y. Christen (eds.), *Temporal Coding in the Brain*, Springer, Berlin, 1994, pp. 51 - 79.
- [6] D. Horn, Networks of complex neurons, *Physica*, **A200**, 594 - 601, 1993.
- [7] W. Gerstner, R. Ritz, and J. L. van Hemmen, Why spikes? Hebbian learning and the retrieval of time-resolved excitation patterns, *Biol. Cybern.*, **69**, 503-515, 1993.
- [8] D. Hansel, H. Sompolinsky, Synchronization and computation in a chaotic neural network, *Phys. Rev. Lett.*, **68**, 718-721, 1992.
- [9] O. Kinouchi, M. H. R. Tragtenberg, Modeling neurons by simple maps, *International Journal of Bifurcation and Chaos*, to appear, 1996.
- [10] K. Kaneko, Relevance of dynamical clustering to biological networks, *Physica*, **D75**, 55-73, 1994.
- [11] K. Aihara, T. Takabe, and M. Toyoda, Chaotic neural networks, *Physics Letters A*, **144**, 333-340, 1990.
- [12] K. Aihara, Chaos in Axons, in: M. A. Arbib, *The Handbook of the Brain Theory and Neural Networks*, MIT Press, Cambridge, 1995, pp. 183-185.
- [13] F. Pasemann, Dynamics of a single model neuron, *Int. J. Bif. and Chaos*, **3**, 271 -278 (1993).
- [14] M. Hirsch, Network dynamics: Principles and problems, in: F. Pasemann, H. D. Doebner, *Neurodynamics*, World Scientific, Singapore, 1991, pp. 3 - 29.

- [15] M. C. Mackey, L. Glass, Oscillation and chaos in physiological control systems, *Science*, **197**, 287-289, 1977.
- [16] W. J. Freeman, Tutorial on neurobiology: From single neurons to brain chaos, *Int. J. Bif. and Chaos*, **2**, 451-482, 1992.
- [17] A. Babloyantz, C. Lourenço, Computation with chaos: A paradigm for cortical activity, *Proc. Natl. Acad. Sci. USA*, **91**, 9027-9031, 1994.
- [18] I. Tsuda, Dynamic link of memory - chaotic memory map in nonequilibrium neural networks, *Neural Networks*, **5**, 313-326, 1992.

Appendix: Additional Material

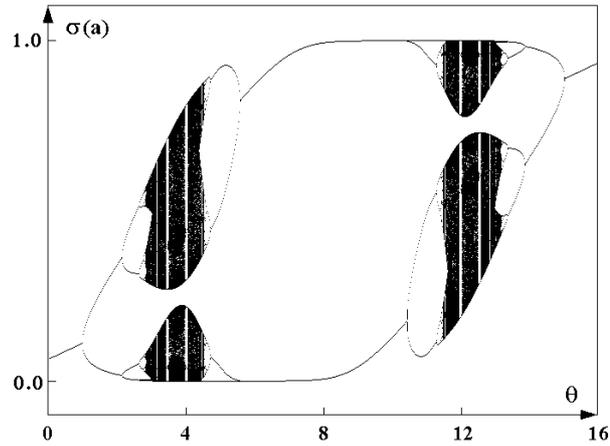


Figure 6: Bifurcation diagram for θ with fixed parameters $\gamma = 0.6$ and $w = -16$.

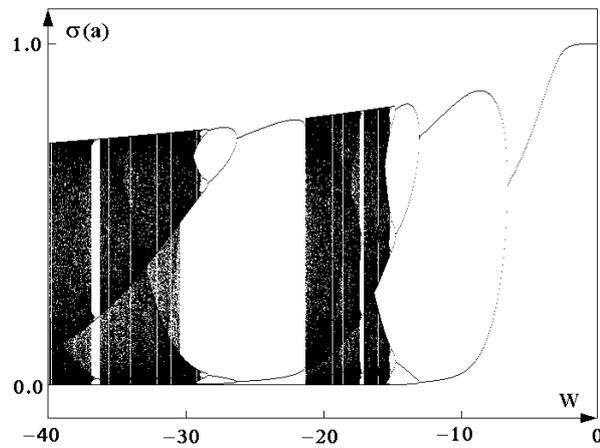


Figure 7: Bifurcation diagram for self-coupling w with fixed parameters $\gamma = 0.6$ and $\theta = 4$.

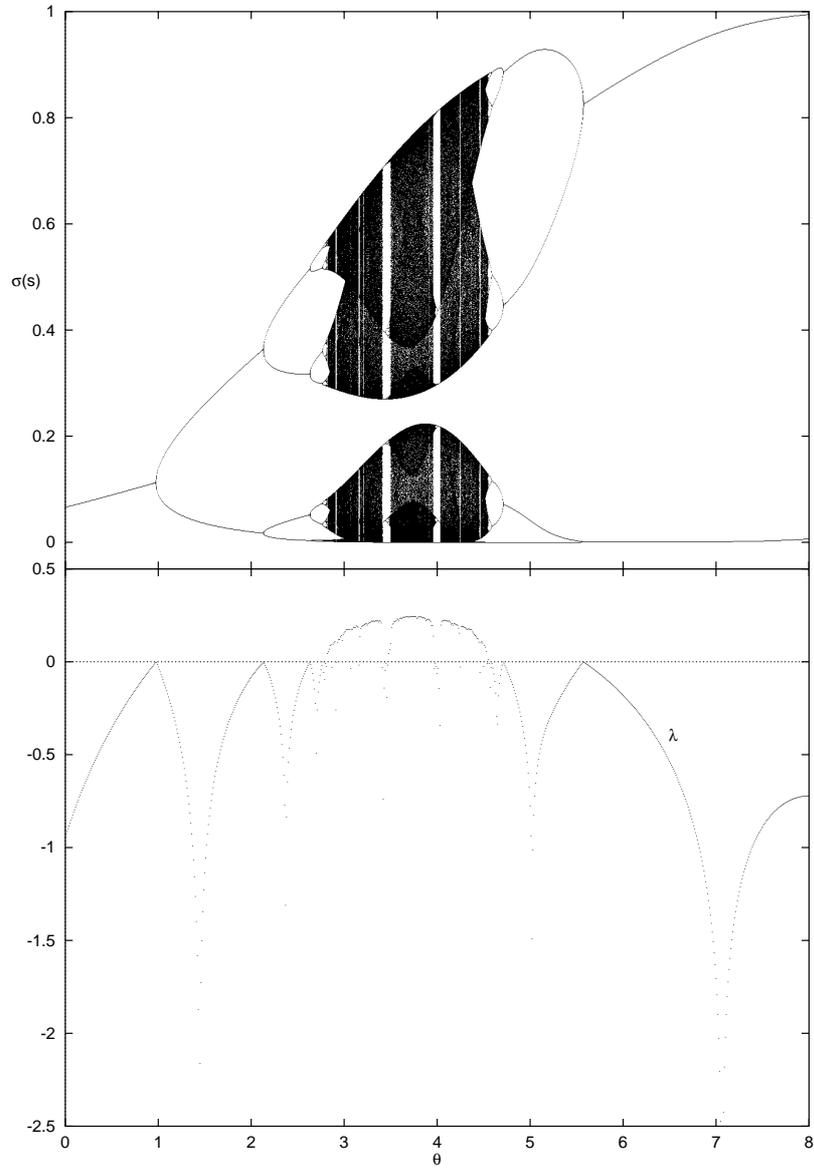


Figure 8: Bifurcation diagram for θ with fixed parameters $\gamma = 0.6$ and $w = -16$. Below the Lyapunov λ exponents are plotted for the same θ -interval. A positive λ indicates the existence of a chaotic attractor.