

Synchronized Chaos in Coupled Neuromodules of Different Types *

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Abstract

We discuss the time-discrete parametrized dynamics of two coupled recurrent neural networks. General conditions for the existence of synchronized dynamics are derived for these systems, and it is demonstrated that also the coupling of totally different network structures can result in periodic, quasiperiodic as well as chaotic dynamics constrained to a synchronization manifold M . Stability of the synchronized dynamics can be calculated by Lyapunov exponent techniques. In general, in addition to synchronized attractors there often co-exist asynchronous periodic, quasiperiodic and even chaotic attractors. Simulation results with respect to a minimal coupling scheme for neuromodules of different type are presented.

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1 Introduction

There is experimental evidence that coherent firing of spatially separate neurons in biological brains appears as a response to specific external stimuli. This led to the famous “binding hypothesis” which states that selective synchronization of neural activity is a fundamental temporal mechanism for binding spatially distributed features into a coherent object. (cf. e.g. [5], [6], [19]). In this context conceptual discussions and biologically motivated models were mainly based on synchronization of oscillatory dynamics in large, e.g. high-dimensional systems. But with respect to brain theory it seems that one is often not aware of the rich phenomenology of coupled nonlinear subsystems.

On this background we will study the parametrized time-discrete dynamics of two coupled neural networks with recurrent connectivity. The subsystems are called *neuromodules* because they are considered as basic building blocks for larger neural networks. These neuromodules are described as low-dimensional dynamical systems with nonlinearities introduced by the sigmoidal transfer functions of standard additive neurons. As parameters we will consider bias terms and/or stationary inputs, the synaptic strength (or weights) between module neurons, and the coupling strength between neurons of different modules.

On the other hand, since 1990 the synchronization of chaotic systems has been extensively investigated experimentally as well as theoretically. Although a large part of the work has been motivated by its potential for technical applications there is still an ongoing discussion of the theoretical aspects of this phenomenon [7], [14], [16], [17], [18]. Most articles study coupled time-continuous systems like Lorenz or Rössler systems; but – as in this contribution – also time-discrete systems are considered (e.g. [3], [4], [8]).

We will use the term “synchronization” here in the sense of *complete synchronization*; i.e. the states of the systems will coincide, while the dynamics in time remains periodic or chaotic. Thus, the coupled networks under consideration will have the same number of neurons. Often it is claimed that complete synchronization appears only if the interacting systems are identical. But the synchronization conditions derived in section II show that synchronization of corresponding neurons can be achieved even if the systems are different; i.e. the coupled networks have different recurrent architectures. Synchronous chaos in the case of bi-directionally coupled identical neuromodules has been discussed e.g. in [11]; for the case of driven neuromodules see [12], for two coupled chaotic neurons [13].

As an example for synchronized dynamics in neuromodules of different types, in section III a three neuron ring network is coupled with a bi-directional chain of three neurons. Although the dynamical features of the isolated systems are quite different – besides fixed point attractors 3-rings can have period-2, -3 and period-6 attractors [9], whereas 3-chains can have p -periodic attractors for all p , and chaotic attractors as well [10]– there are many different coupling schemes

which guarantee the existence of completely synchronized dynamics. Computer simulations demonstrate stable synchronous chaos for the case of a “minimal” coupling scheme. General aspects of synchronizing neuromodules are discussed in section IV.

2 Coupled neuromodules

We are considering a neuromodule with n units as a discrete parametrized dynamical system on an n -dimensional activity phase space \mathbf{R}^n . With respect to a set ρ of parameters it is given by the map $f_\rho : \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by

$$a_i(t+1) = \theta_i + \sum_{j=1}^n w_{ij} \sigma(a_j(t)), \quad i = 1, \dots, n, \quad (1)$$

where $a_i \in \mathbf{R}^n$ denotes the activity of the i -th neuron, and $\theta_i = \bar{\theta}_i + I_i$ denotes the sum of its fixed bias term $\bar{\theta}_i$ and its stationary external input I_i . The output $o_i = \sigma(a_i)$ of a unit is given by the standard sigmoidal transfer function $\sigma(x) := (1 + e^{-x})^{-1}$, $x \in \mathbf{R}$, and w_{ij} denotes the synaptic weight from unit j to unit i . If there exists a parameter set $\rho = (\theta, w)$ for which the dynamics (1) has at least one chaotic attractor, the module will be called a *chaotic neuromodule*.

Now, let A and B denote two neuromodules (1) with parameter sets $\rho^A = (\theta^A, w^A)$ and $\rho^B = (\theta^B, w^B)$, respectively. Connections going from module B to module A are given by an $(n \times n)$ -coupling matrix w^{AB} . Correspondingly, connections from module A to module B are given as a matrix w^{BA} . Thus, the architecture of the $2n$ -dimensional coupled system is given by a matrix w of the form

$$w = \begin{pmatrix} w^A & w^{AB} \\ w^{BA} & w^B \end{pmatrix}. \quad (2)$$

The neural activities of module A and B will be denoted a_i, b_i , $i = 1, \dots, n$, respectively; and $F_\rho : \mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}$ denotes the parametrized dynamics of the coupled systems with respect to $\rho := (\rho^A, \rho^B, w^{AB}, w^{BA})$.

In the following we will consider the process of *complete synchronization*, which means that there exists a subset $D \subset \mathbf{R}^{2n}$ such that $(a_0, b_0) \in D$ implies

$$\lim_{t \rightarrow \infty} |a(t; a_0) - b(t; b_0)| = 0,$$

where $(a(t; a_0), b(t; b_0))$ denotes the orbit under F_ρ through the initial condition $(a_0, b_0) \in \mathbf{R}^{2n}$. Thus we are interested in the case where corresponding neurons of the modules have identical activities during a process. A *synchronized state* s of the coupled system is defined by $s := a = b \in \mathbf{R}^n$, and the *synchronization manifold* $M := \{(s, s) \in \mathbf{R}^{2n} \mid s = a = b\}$ of synchronized states corresponds to an n -dimensional hyperspace $M \cong \mathbf{R}^n \subset \mathbf{R}^{2n}$.

A straight forward calculation will prove the following general synchronization condition:

Lemma 1 *Let the parameter sets ρ^A , ρ^B of the modules A and B satisfy*

$$\theta^A = \theta^B, \quad (w^A - w^{BA}) = (w^B - w^{AB}). \quad (3)$$

Then every orbit of F_ρ through a synchronized state $s \in M$ is constrained to M for all times; i.e. M is an F_ρ -invariant manifold.

The condition (3) shows that synchronization can be achieved for modules with different weight matrices w^A and w^B , as well as with different coupling matrices w^{AB} and w^{BA} , as long as (3) is satisfied. Using the definitions $\theta := \theta^A = \theta^B$ and

$$w_{ij}^+ := (w_{ij}^A + w_{ij}^{AB}) = (w_{ij}^B + w_{ij}^{BA}), \quad (4)$$

the corresponding synchronized dynamics $F_\rho^s : M \rightarrow M$ is then given by the n equations

$$s_i(t+1) = \theta_i + \sum_{j=1}^n w_{ij}^+ \cdot \sigma(s_j(t)), \quad i = 1, \dots, n. \quad (5)$$

Thus, the synchronized dynamics corresponds to that of an n -module with weight matrix w^+ , and it depends on the choice of the coupling matrices w^{AB} and w^{BA} satisfying (3).

Although the persistence of the synchronized dynamics is guaranteed by condition (3), it is not at all clear that the dynamics constrained to the manifold M is asymptotically stable with respect to the dynamics F_ρ ; i.e. if a small perturbation of the system in a synchronous mode will desynchronize the system or not. A periodic or chaotic orbit in M may be an attractor for the synchronized dynamics F_ρ^s but not for the dynamics F_ρ of the coupled system [2]. We therefore have to discuss stability aspects of the synchronized dynamics with the help of Lyapunov exponents, and will discern between *synchronization exponents* λ_i^s and *transversal exponents* λ_i^\perp , $i = 1, \dots, n$ (compare e.g. [11]). They are derived from the linearizations $L^+(s(t))$ and $L^-(s(t))$, respectively, of the systems dynamics F_ρ along synchronized orbits $s(t)$ constrained to M ; for $i, j = 1, \dots, n$ we have

$$L_{ij}^+(s) := w_{ij}^+ \cdot \sigma'(s_j), \quad L_{ij}^-(s) := w_{ij}^- \cdot \sigma'(s_j), \quad (6)$$

with σ' denoting the derivative of the sigmoid σ , w^+ as in equation (4), and w^- given by

$$w_{ij}^- := (w_{ij}^A - w_{ij}^{BA}) = (w_{ij}^B - w_{ij}^{AB}). \quad (7)$$

Synchronized chaotic dynamics will be characterized by the largest synchronization exponent satisfying $\lambda_1^s > 0$. On the other hand, a positive transversal exponent λ_1^\perp indicates unstable synchronized dynamics. Thus, if an unstable synchronous chaotic orbit exists in M then the system naturally must have entered a *hyperchaotic* regime [15]; i.e. at least two Lyapunov exponents of the system F_ρ are positive.

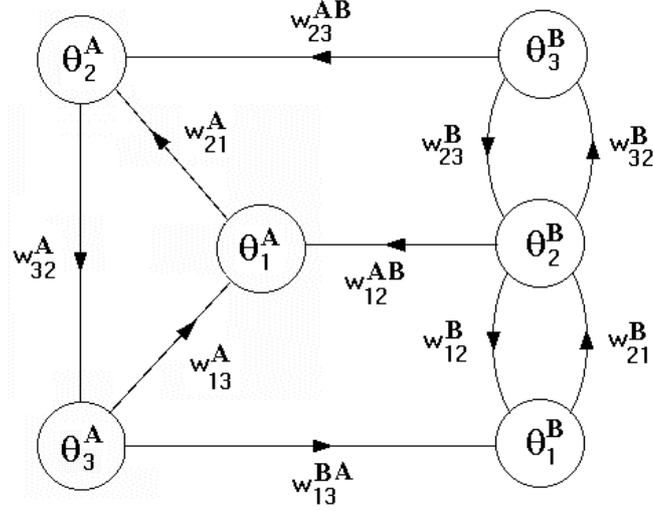


Figure 1: A minimal coupling configuration for complete synchronization of a 3-ring (module A) with a bidirectional 3-chain (Module B).

3 Example: Coupled 3-neuron modules

To demonstrate the complete synchronization of two different types of networks, we will study the following setup where an (oscillatory) 3-ring is coupled to a (chaotic) bi-directional 3-chain. The modules and their couplings are shown in figure 1, and the dynamics of the coupled system is given by

$$\begin{aligned}
 a_1(t+1) &:= \theta_1^A + w_{13}^A \sigma(a_3(t)) + w_{12}^{AB} \sigma(b_2(t)) , \\
 a_2(t+1) &:= \theta_2^A + w_{21}^A \sigma(a_1(t)) + w_{23}^{AB} \sigma(b_3(t)) , \\
 a_3(t+1) &:= \theta_3^A + w_{32}^A \sigma(a_2(t)) , \\
 b_1(t+1) &:= \theta_1^B + w_{12}^B \sigma(b_2(t)) + w_{13}^{BA} \sigma(a_3(t)) , \\
 b_2(t+1) &:= \theta_2^B + w_{21}^B \sigma(b_1(t)) + w_{23}^B \sigma(b_3(t)) , \\
 b_3(t+1) &:= \theta_3^B + w_{32}^B \sigma(b_2(t)) .
 \end{aligned}$$

A possible realization of the synchronization condition (3) for this special case together with the corresponding parameter values reads

$$\begin{aligned}
 w_{12}^B = w_{12}^{AB} = 8 & \quad , \quad w_{13}^A = w_{13}^{BA} = -8 , \\
 w_{21}^A = w_{21}^B = 8 & \quad , \quad w_{32}^A = w_{32}^B = 8 , \\
 w_{23}^B &= w_{23}^{AB} = -8 , \\
 \theta_1 = -1 , \quad \theta_2 &= -3.6 , \quad \theta_3 = -4 .
 \end{aligned}$$

For these parameter values the coupled system has a stable synchronized 2-cyclic chaotic attractor, which is depicted in figure 2. Recall [1], that a chaotic attractor

is called *p-cyclic* if it has p connected components which are permuted cyclically by the map F_ρ ; i.e. every component of a p -cyclic attractor is an attractor of F_ρ^p . Figure 2 shows the projections of the chaotic attractor onto the (o_1^A, o_2^A) -subspace and onto the subspaces (o_1^A, o_1^B) , (o_2^A, o_2^B) , and (o_3^A, o_3^B) , demonstrating that the chaotic orbit is in fact constrained to M (main diagonal in the last three subspaces).

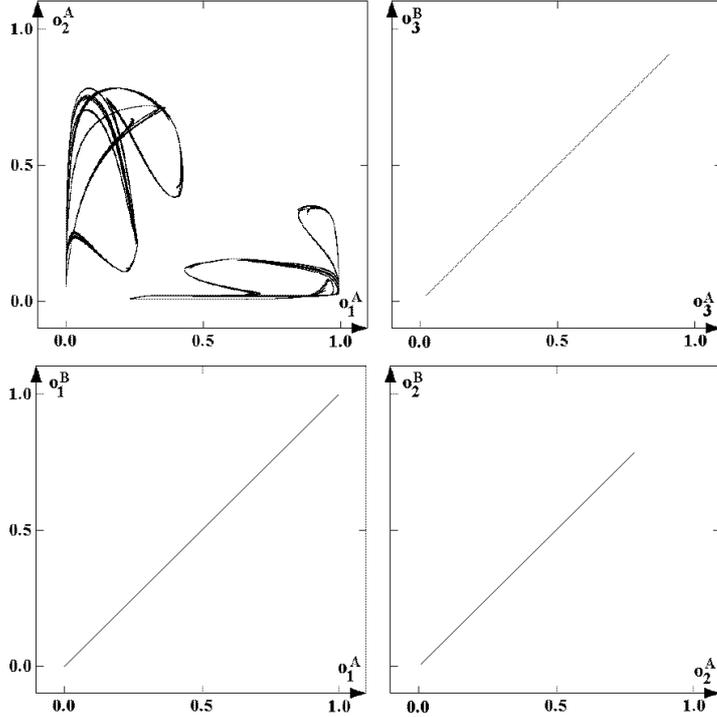


Figure 2: Completely synchronized chaos for two coupled 3-modules having different architectures. Parameters: see text.

Furthermore, we simulated the system with parameters from a neighbourhood of the values given above. The completely synchronized 3-dimensional dynamics of this coupled system is given by equation (5). That it has interesting dynamical features can be read from the bifurcation sequence with respect to the variation of $\theta_2 := \theta_2^A = \theta_2^B$ shown in figure 3. The parameters are here given by $w_{12}^+ = w_{21}^+ = w_{32}^+ = -w_{23}^+ = -w_{13}^+ = 8$, $\theta_1 = -1$, $\theta_3 = -4$. Starting from a fixed point attractor for $\theta_2 = -8$, the system jumps into a period doubling route to chaos followed by windows of periodic, quasi-periodic and chaotic dynamics. There are also θ_2 -intervals where we observe coexisting synchronous periodic and chaotic attractors. In fact, for this special coupled system the synchronized dynamics will always be stable; this can be easily seen by the fact that the only nonzero elements of w^- in (7) are w_{21}^- and w_{32}^- , and thus the matrices $L^-(s)$ in (6) have zero eigenvalues for all synchronous states s . Therefore the largest transversal Lyapunov exponent λ_1^\perp will be negative for all orbits constrained to M .

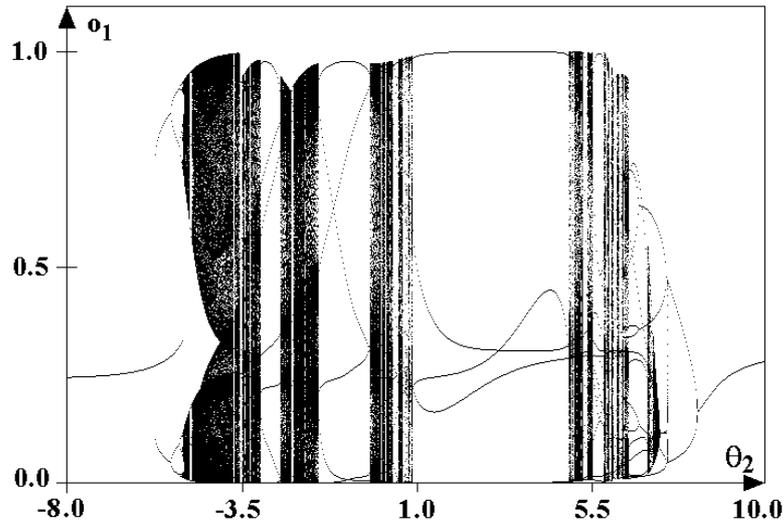


Figure 3: Bifurcation sequence for the completely synchronized dynamics of coupled 3-modules. Parameters: see text.

4 Conclusions

It has been shown that also in systems of two coupled neuromodules with the same number of neurons but *different* architectures synchronous chaos as well as synchronized periodic or quasiperiodic dynamics can exist. The configuration discussed in section III is a special case in the sense that the coupling structure is *minimal*. This had the effect that the synchronized dynamics is stable for all parameter values. Thus, even coexisting attractors are always constrained to the synchronization manifold M . Introducing only one more coupling connection, w_{13}^{AB} for instance, satisfying the synchronization condition (3), will generate all the phenomena described e.g. in [11]; i.e. depending on module parameters, orbits constrained to a synchronization manifold can be globally or locally stable, or unstable. For large parameter domains stable synchronous dynamics will coexist with asynchronous periodic, quasiperiodic or even chaotic attractors. Thus, whether a system ends up asymptotically in a synchronous mode or not depends crucially on initial conditions, i.e. on the *history* of the system itself. This may introduce *memory effects* into the behavior of coupled systems.

The conditions (3) for the existence of complete synchronization require that the sum of bias terms and stationary external inputs of corresponding module neurons are identical. A synchronized mode of the coupled system persists even if parameters, like corresponding external inputs, are varying slowly. Thus, the synchronized dynamics may pass through a whole bifurcation sequence, and this can be understood as a sign for time-varying input signals with amplitudes having a fixed ratio (recall that inputs may correspond to the weighted outputs of other

units of a larger system).

Analysis as well as computer simulations show, that de-synchronization of module dynamics can be achieved in various ways: Diverging external inputs or other diverging parameters (like module weights or the strength of couplings between modules) will immediately de-synchronize the modules.

The presented results can stimulate new dynamical models for networks with higher information processing (or cognitive) capabilities. The rather typical co-existence of synchronized modes with modes of asynchronous dynamics relativizes functional properties (like “feature binding”) attributed to synchronization, but at the same time introduces memory aspects into these systems through generalized hysteresis effects. Furthermore, since synchronization and de-synchronization of modules can be controlled by different parameters, attention guided synchronization of subsystems is an additional interesting functional feature of coupled neuromodules.

References

- [1] Abraham, R. H., Gardini, L. and Mira, C. (1997), *Chaos in Discrete Dynamical Systems*, Springer-Verlag, New York.
- [2] Ashwin, P., Buescu, J., and Stewart, I. (1996) From attractor to chaotic saddle: a tale of transverse instability, *Nonlinearity*, **9**, 703–737.
- [3] Astakhov, V., Hasler, M., Kapitaniak, T., Shabunin, A., and Anishchenko, V. (1998) Effect of parameter mismatch on the mechanism of chaos synchronization loss in coupled systems, *Phys. Rev. E*, **58**, 5620–5628.
- [4] De Sousa Viera, M., Lichtenberg, A. J. and Liebermann, M.A. (1992) Synchronization of regular and chaotic systems, *Phys. Rev. A*, **46**, 7359–7362.
- [5] Eckhorn, R., Bauer, R., Jordan, W., Brosch, M., Kruse, W., Munk, M. and Reitboeck H. J. (1988) Coherent oscillations: a mechanism for feature linking in the visual cortex. *Biol. Cybern.*, **60**, 121-130.
- [6] Engel, A. K., Roelfsema, P. R, Fries, P., Brecht, M. and Singer, W. (1997) Binding and response selection in the temporal domain - a new paradigm for neurobiological research? *Theory in Biosciences*, **116**, 241-266.
- [7] Hasler, M., Maistrenko, Y., and Popovych, O. (1998) Simple example of partial synchronization of chaotic systems, *Phys. Rev. E*, **58**, 6843–6846.
- [8] Maistrenko, Y., and Kapitaniak, T. (1996) Different types of chaos synchronization in two coupled piecewise linear maps, *Phys. Rev. E*, **54**, 3285–3292.

- [9] Pasemann, F. (1995), Characteristics of periodic attractors in neural ring networks, *Neural Networks*, **8**, 421-429.
- [10] Pasemann, F. (1995), Neuromodules: A dynamical systems approach to brain modelling. In Herrmann, H., Pöppel, E. and Wolf, D. (eds.), *Supercomputing in Brain Research - From Tomography to Neural Networks*, Singapore: World Scientific, pp. 331-347.
- [11] Pasemann, F. (1999), Synchronous and asynchronous chaos in coupled neuromodules, *International Journal of Bifurcation and Chaos*, **9**.10, to appear.
- [12] Pasemann, F. (1999), Driving neuromodules into synchronous chaos, in: Proceedings, IWANN'99, Alicata, June 2-4, 1999, LNCS, Springer Verlag, Berlin, to appear.
- [13] Pasemann, F. (1999) Synchronized chaos and other coherent states for two coupled neurons, *Physica D*, **128**, 236-249.
- [14] Pikovsky, A. S., Rosenblum, M. G., Osipov, G.V., and Kurths, J. (1997) Phase synchronization of chaotic oscillators by external driving, *Physica D*, **104**, 219-238.
- [15] Rössler, O. (1979) An equation for hyperchaos. *Phys. Lett. A*, **71**, 155-157.
- [16] Rosenblum, M. G., Pikovsky, A. S., and Kurths, J. (1996) Phase synchronization of chaotic oscillators, *Phys. Rev. Lett.*, **76**, 1804-1807.
- [17] Rulkov, N. F., Sushchik, M. M., Tsimring, L. S., and Abarbanel, H. D. I. (1995) General synchronization of chaos in directionally coupled chaotic systems, *Phys. Rev. E*, **51**, 980-994.
- [18] Schweizer, J., Kennedy, M. P., Hasler, M. and Dedieu, H. (1995) Synchronization theorem for a chaotic system. *Int. J. Bifurcat. Chaos*, **5**, 297-302.
- [19] Singer, W. (1994) Time as coding space in neocortical processing. In: Buzsáki, G.; Llinás, R.; Singer, W.; Berthoz, A. and Christen, Y. (eds.) *Temporal Coding in the Brain*. Springer, Berlin. pp. 51-80.